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Jump-Diffusion Parameters and Passage Times Estimation

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Abstract

The main purposes of this paper are two contributions: (1) it presents a method, which is the first passage time generalized for all passage times (GPT method), in order to estimate the parameters of stochastic Jump-Diffusion process. (2) it compares in a time series model, share price of gold, the empirical results of the estimation and forecasts obtained with the GPT method and those obtained by the moments method and the FPT method applied to the Merton Jump-Diffusion (MJD) model.

Keywords and phrases Merton Jump-Diffusion; First passage time; Black-Scholes equation; Trajectory.
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1 Introduction

The parameters estimation is one of main dynamic models problems in many scientific fields, particularly in economics and finance. In the reference model proposed in 1973 by Black-Scholes some assumptions (constant volatility, log-normality of returns, continuities of trajectories...) are required. This process is known as the geometric Brownian. A number of empirical observations clearly contradict these assumptions . Now, many different models are proposed ([6],[5]) to modify the Black- Scholes model. To solve the problems associated with the Black-Scholes model, Merton, in 1976, introduced a new financial model by using the discontinuities by a Poisson process with Gaussian jumps. We take MJD model to estimate the parameters using the method of moments. We then present and use the GPT method to estimate the same parameters and we finally compare the empirical results obtained by the different methods on a time series model: gold share price. Our sample comprises observations data of gold share price (US$ by once in London Bullion Market) and observations started on January 2^{nd}, 2007 to October 31^{st}, 2007 covering a period of 212 days.

2 Merton jump diffusion model

We consider that the assiet price S_t presents log-normal jumps $V_1, ..., V_j$ at random times $\tau_1, ..., \tau_j$, which represent the moments of jumping of a Poisson process (Bandi [2] , Kou [5] ,). Between two instants, we assume that the dynamics of the model follows the Black-Scholes process model. It is a continuous time model. We suppose that the behavior of the stock price is determined by the stochastic differential equation :

$$dS_t = \mu S_t dt + \sigma S_t dW_t, S_0 = x_0 \tag{1}$$

where W is a standard Brownian motion and μ and σ are respectively the drift and diffuse coefficient.

In the MJD model, the price process S_t is assumed to follow the stochastic differential equation

$$\frac{dS_t}{S_{t^-}} = \mu dt + \sigma dW_t + d\left(\sum_{j=0}^{N_t} (V_j - 1)\right) \tag{2}$$

where $\log V_j \sim i.i.d.N\left(\alpha,\ \delta^2\right)$ .The last term models the jumps. A jump is modelled by the random variable V which transforms the price S_t to VS_t. The difference $(V-1)$ is the relative change in price when a Poisson jump occurs.Using Ito'lemma, the solution of (2) is

$$S_t = S_0 \exp\left\{\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W_t\right\}\prod_{j=0}^{N_t} V_j \tag{3}$$

where S_0 is the asset price at time zero and with $Y_j = \log V_j$ we can write

$$X_t = \log\frac{S_t}{S_0} = \left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W_t + \sum_{j=0}^{N_t} Y_j \tag{4}$$

The processes W, N and the random variables $Y_j \sim N\left(\alpha,\ \delta^2\right)$ are supposed to be independent.

The discontinuities of the price process are described by the Poisson process N with intensity λ (mean arrival rate of jumps per unit time) and jump V_j. Introduction of the MJD model adds three extra parameters $\left(\alpha,\ \delta^2,\ \lambda\right)$ to the Black-SCholes process model which contains two parameters $\left(\mu,\ \sigma^2\right)$. We consider that the jumps are symmetrical and with mean value nul. The probability density of ΔX_t can be expressed (Askari and Krichene [1])

$$f(x) = \sum_{n=0}^{\infty} e^{-\lambda}\frac{\lambda^n}{n!}\left[\frac{1}{\sqrt{2\pi\left(\sigma^2 + n\delta^2\right)}}\exp\left(-\frac{\left(x - \left(\mu - \frac{\sigma^2}{2}\right)\right)^2}{2\left(\sigma^2 + n\delta^2\right)}\right)\right] \tag{5}$$

3 Parameters estimation

3.1 Method of Moments

Our model is described by four parameters μ, σ^2, λ and δ^2 and we will calculate the estimators by the moments method which was used by Beckers [3]. The idea was based on the equalization of four empirical moments with the corresponding theorical central moments. This lead to solve a system of four equations that alowed to determine the estimators [1]. Given the law of returns X, the central moments of odd order are nul and the central moments of even order can be written

$$E\left(\left(X - E\left(X\right)\right)^{2k}\right) = \frac{(2k)!}{2^k k!}\sum_{n=0}^{\infty} e^{-\lambda}\frac{\lambda^n}{n!}\left(\sigma^2 + n\delta^2\right)^k \tag{6}$$

One way to determine the values of the parameters μ, σ^2, λ and δ^2 is to fit the market data, is to solve the following system of equations

$$\begin{cases} E(X) = \mu - \frac{\sigma^2}{2} \\ E\left(\left(X - E\left(X\right)\right)^2\right) = \sigma^2 + \lambda\delta^2 \\ E\left(\left(X - E\left(X\right)\right)^4\right) = 3\sum_{n=0}^{\infty} e^{-\lambda}\frac{\lambda^n}{n!}\left(\sigma^2 + n\delta^2\right)^2 = 3\left(\left(\sigma^2 + \lambda\delta^2\right)^2 + \lambda\delta^4\right) \\ E\left(\left(X - E\left(X\right)\right)^6\right) = 15\sum_{n=0}^{\infty} e^{-\lambda}\frac{\lambda^n}{n!}\left(\sigma^2 + n\delta^2\right)^3 \\ \qquad = 15\left(\left(\sigma^2 + \lambda\delta^2\right)^3 + 3\lambda\delta^4\left(\sigma^2 + \lambda\delta^2\right) + \lambda\delta^6\right) \end{cases} \tag{7}$$

3.2 FPT method

This method proposes the parameters estimation of the process X by the observation of the first passage time T ([5]), by the line of equation $X = S$. Janssen and al used the first passage time method to estimate the two parameters of the stochastic differential equation (Black-Scholes equation). This method consists of determining a constant boundary limit S after having decomposed the original time-series in several (k) independent series having the same initial point $x_0 = x_{1,0} = ... = x_{k,0}$ where $x_{i,0}$ $(i = 1, ..., k)$ denotes the initial point of the series number i. This constant terminal S intersects all the k trajectories. Each first intersection (with the trajectory i) determines a moment t_i (called first passage time). The random variable T is the first passage time of the process $X(t) = \exp(Y(t))$ by the point $S = \exp(a)$. We have : $T = \inf\{t/X(t) = S,\ t > 0\}$. The first time T follows, as proved by Chhikara and Al. [4] and Folks and Al. [8], an inverse gaussian distribution, and its density function, with $X(t_0) = x_0$, is :

$$f(S, t/x_0, t_0) = \frac{\log\left(\frac{S}{x_0}\right)}{\sqrt{2\pi}\sigma(t-t_0)^{\frac{3}{2}}} \exp\left\{-\frac{\left[\log\left(\frac{S}{x_0}\right) - (\mu - \frac{1}{2}\sigma^2)(t-t_0)\right]^2}{2\sigma^2(t-t_0)}\right\} \tag{8}$$

In finance, the trajectory wich is composed of n observations of the stock price is unique and to apply this method we consider several - k - trajectories from only one. The random variables $T_1, ..., T_k$ are the moments of the first passage time by the constant S in the k trajectories , with observed values $t_i : i = 1, ..., k$.

Solving the maximum likelihood equations (8) gives the following estimators :

$$\begin{cases} \widehat{\mu} = \frac{\log\left(\frac{S}{x_0}\right)}{\bar{t}} + \frac{1}{2}\widehat{\sigma}^2 \\ \widehat{\sigma}^2 = \frac{\left[\log\left(\frac{S}{x_0}\right)\right]^2}{k} \sum_{i=1}^{k}\left(\frac{1}{t_i} - \frac{1}{\bar{t}}\right) \end{cases} \tag{9}$$

where $\bar{t} = \sum_{i=1}^{k}\frac{1}{t_i}$.

In our work we study, in the first step, the quality of the estimations. We solve this problem by adopting an algorithm (from the simulation of uniform law on the variation intervals of S) to determine the best estimators $\widehat{\mu}$ and $\widehat{\sigma}^2$, according to certain criteria (RMSE, %RMSE and %RME) by comparaison with the initial series. The chosen couple is the one which minimizes the optimization criteria.

$i)$ We consider the k independent trajectories, each with n_i observations $(i = 1, ..., k)$ and we define :

n_i : observations number of the trajectory i, with ${}_{i=1}^{k} n_i = n$.

x_{ij} : observation number j of the trajectory number i, $(i = 1, ..., k;\ j = 1, ..., n_i)$.

t_{ij} : observation number moment j of the trajectory number i.

3.3 GPT method

In the second step, we generalize this method. we use all passage times crossed by the value S so that, this value is reached by all possible trajectories. This method principle is similar to that of the FPT method. The random variable T follows an inverse gaussian law. Its density function is written in the form (8): Solving the maximum likelihood equations (8) gives the following estimations :

$$\begin{cases} \widehat{\mu} = \frac{\log\left(\frac{S}{x_0}\right)}{\bar{t}} + \frac{1}{2}\widehat{\sigma}^2 \\ \widehat{\sigma}^2 = \frac{\left[\log\left(\frac{S}{x_0}\right)\right]^2}{m} \sum_{i=1}^{m}\left(\frac{1}{t_i^l} - \frac{1}{\bar{t}}\right) \end{cases} \tag{10}$$

with $\bar{t} = \frac{1}{m}\sum_{i=1}^{k}\sum_{l=1}^{m_k} t_i^l$ and where $m = \sum_{l=1}^{k} m_l$ is the total number of passage times, m_l is the number of passage times in the trajectory number l and t_i^l is the passage time number l in the trajectory number i.

The random variables $T_1, ..., T_m$ are the moments of the passage time by the line $X = S$ in the k trajectories, with observed values $t_i^l : i = 1, ..., k; \; l = 1, ..., m_k$.

4 Empirical Results

4.1 Observations

We used the 212 daily observations of share price of gold to calculate the returns $X_t = \log \frac{S_{t+1}}{S_t}$, $t = 0, 1, ...211$. We obtain an i.i.d. sample with which we calculate the empirical estimators of the necessary four moments: the empirical expectation and the first moments. This observations are represented in figure 1. We note that our data are not gaussian. Otherwise the Black-Scholes model is sufficiently robust. The normality test of Jarque Bera gives a $P-$value equals to $46.0063 > \chi^2_{1-\alpha} = 5.99$ for $\alpha = 0.05$. This leads to reject the nul hypothesis, i.e., the law of return can not be a normal distribution.

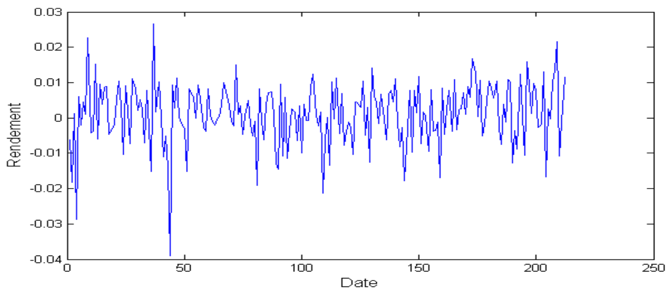

Figure 1 Returns evolution.

4.2 Estimations

FPT method: In our time-series the initial value $x_o = 608.30$ divises the trajectory in tWO $(b = 2)$ trajectories and by random simulation we find $S = 640.75$ such as all the 2 trajectories are reached by this value S. The numerical results are : $\widehat{\mu} = 3.3327 \times 10^{-4}$ and $\widehat{\sigma}^2 = 0.0097$.

GPT method: $x_o = 608.30$ We find by random simulation $S = 640.75$ such as all the trajectories are reached by this value S. All passage times are considered. There are $(n = 4)$. We find the estimations : $\widehat{\mu} = 0.0021$ and $\widehat{\sigma}^2 = 6.5117 \times 10^{-4}$

Méthode des moments

La paramétrisation selon le modèle de Merton, résolvant selon la méthode des moments le système d'équations précédent, donne les résultats suivants :

$\widehat{\mu} = 1.0288 \times 10^{-3}$, $\widehat{\sigma}^2 = 4.8731 \times 10^{-5}$, $\lambda = 0.244$, $\delta^2 = 1.2361 \times 10^{-4}$.

We takes the estimated parameters $\widehat{\mu}$ and $\widehat{\sigma}^2$ by the three methods for the simulation (see fig.2) and obtains, table 1 :

	RMSE	%RMSE	ERM(%)
FPT method	19.0749	0.0273	2.7327
GPT method	17.959	0.0271	2.171
Moment method	22.891	0.0327	2.743

Table 1 Simulation Errors.

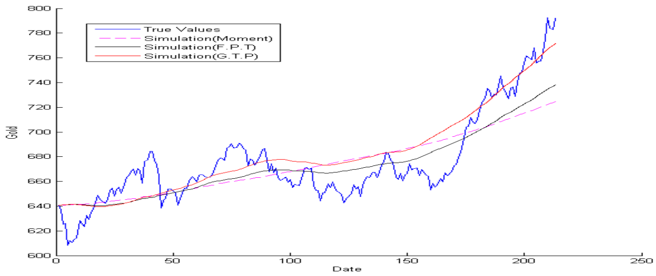

Figure 2 Simulations and true values.

4.3 Prevision

The forecast days is 10 days. The graphs of the forecast errors for each method are represented in figure 3. The result analysis show that the GPT method maxerror does not exceed 7%.

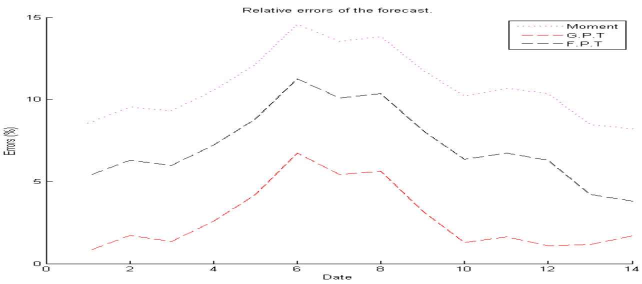

Figure 3 Relative errors.

5 Conclusion

In this paper, we evaluate the performance of a method in the parameters estimation problem of MJD model. According to the simulation and forecast results, we deduce that this GPT method gives better results than the method of moments and the FPT method and can be used in other problems based on the stochastic differential equations with jumps.

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