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Structure and inference in high-order self-excited multiple thresholds $GINAR$ models

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Abstract

We propose a threshold integer-valued autoregressive model, with multiple regimes ($K \geq 2$), based on generalized thinning operator (hereafter referred to as $SET-GINAR(K;p)$). First, we study the probabilistic structure of our model through the stationarity issue and moments structure. Second, we provide three statistical inference procedures, namely: two estimation methods, as well as a nonlinearity test procedure to test the threshold effect. Finally, the performances of the obtained inference procedures will be evaluated via an intensive simulation study and application on real data.

Keywords and phrases Count integer-valued process, $SET-GINAR(K;p)$ model, CLS and CML estimators, nonlinearity time series.

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1 Motivations and formulations

Numerous count time series encountered in various fields reveal strong nonlinearity characteristics that cannot be captured by the standard linear count time series models like $INAR$-type schemes (Al-Osh and Alzaid (1987); Du and Li (1991)). This nonlinearity fact was emphasized in several works in the literature, we can cite: (Doukhan *et al* (2006). Since nonlinear behavior is often considered to be piecewise linear, this led to the apparition of several works to model the piecewise phenomenon of non-negative integer-valued time series, we have: Thyregod *et al* (1999), Monteiro *et al* (2012), Yang *et al* (2018), Bentarzi and Sadoun (2021), Yang *et al* (2022). All these attractive reasons around the ($SETINAR$) model, motivated us to study the probabilistic structure, the estimation problem as well as the nonlinearity testing issue within a new proposed formulation incorporating multiple thresholds generating more than two regimes. Indeed, we consider a higher-order self-exciting threshold model, with multiple regimes K, ($K > 2$), based on generalized (Latour (1997)) thinning operator, including unspecified innovation distribution but belonging to a parametric family which satisfies only some general technical assumptions. The definition of self–exciting threshold $INAR$ model of order p in each regimes ($SET-GINAR(K,p)$), is given by the following recursion:

$$y_t = \sum_{j=1}^{K} \left[\left(\sum_{i=1}^{p} \varphi_{S_t,i} * y_{t-i}\right) + \varepsilon_{S_t,t}\right] \mathbf{1}\left(S_t = j\right), \ t \in \mathbb{Z} \tag{1.1a}$$

where $\mathbf{1}(.)$ denotes the indicator function, and $\{S_t\}_{t\in\mathbb{Z}}$ is a sequence of independent and identically distributed random variables defined by pieces in time, which is defined as follows:

$$S_t = \begin{cases} 1, & if\ y_{t-d} \leq c_1, \\ 2, & if\ c_1 < y_{t-d} \leq c_2. \\ \vdots & \vdots \\ K-1 & if\ c_{K-2} < y_{t-d} \leq c_{K-1} \\ K & if\ y_{t-d} > c_{K-1} \end{cases}$$

where $(c_1, c_2, ..., c_{K-1})'$ stand the vector of unknown threshold points, and d indicates the delay parameter (such as the delay parameter d is defined like in Tong (1980)). Furthermore, it is assumed that y_{t-1} and

S_t are independent. Thus, we can consider the following notations to associate the belonging in regimes by regions in $\mathbb{Z}_+$:

$$R_1=\{y_{t-d}\in[0,c_1]\}\,;R_j=\{y_{t-d}\in\,]c_{j-1},c_j]\}\,,j=2,\dots,K-1;R_K=\{y_{t-d}\in\,]c_{K-1},+\infty[\}\,.$$

The innovation process, $\{\varepsilon_{j,t},t\in Z,j=1,\dots,K\}$, is a sequence of independent non-negative integer-valued random variables, with some discrete distribution belonging to the parametric family $\{\mathbb{G}_{\underline{\alpha}_j}|\underline{\alpha}_i=(\alpha_{j,1},\alpha_{j,2},\dots,\alpha_{j,q})'\in A\subset\mathbb{R}_+^q\}$, where A is an open, convex subset of $\mathbb{R}_+^q$. For $t\in Z$ and $j=1,\dots,K$, the innovation process $\{\varepsilon_{j,t},t\in\mathbb{Z}\}$ is supposed to be independent of y_{t-i} and $\varphi_{j,i}\circ y_{t-i}$ $\forall i=1,...,p$. The innovation processes $\{\varepsilon_{j,t},t\in\mathbb{Z}\}_{j\in\{1,\dots,K\}}$ are mutually independent. The symbol " $*$ " indicates the generalized thinning operator, which is introduced by Latour (1997) as follows:

$$\varphi_{j,i}*y_{t-i}=\begin{cases}\sum_{k=1}^{y_{t-i}}Y_{k,t,j}^{(i)}, & \textit{if } y_{t-i}>0,\\ 0, & \textit{if } y_{t-i}=0.\end{cases}\qquad(1.1b)$$

Here $\left\{Y_{k,t,j}^{(i)},k\in\mathbb{N},t\in\mathbb{Z},j=1,\dots,K,i=1,...,p\right\}$ are $(i.i.d)$ counting variables with finite mean $\varphi_{j,i}$ $\varphi_{j,i}\in(0,1)\,,j=1,\dots,K$ and $i=1,\dots,p$, which are independent of the innovation process $\{\varepsilon_{j,t},t\in Z,j=1,\dots,K\}$. The sequences of $(i.i.d)$ random variables of counts $\left\{Y_{k,t,j}^{(i)}\right\}_{k\in\mathbb{N},t\in\mathbb{Z},j=1,\dots,K,i=1,\dots,p}$ are mutually independents.

The following notations are used all over the paper:

Let $\underline{\varphi}=\left(\underline{\varphi}_1',\dots,\underline{\varphi}_K'\right)'\in(0,1)^{Kp}$, and $\underline{\alpha}=(\underline{\alpha}_1',\dots,\underline{\alpha}_K')'\in A^K$ with $\underline{\varphi}_j=(\varphi_{j,1},...,\varphi_{j,p})'\in(0,1)^p$, and $\underline{\alpha}_j=(\alpha_{j,1},\dots,\alpha_{j,q})'\in A$, for $j=1,\dots,K$. Defining the $K(p+q)$ dimensioned parameter vector

$$\begin{aligned}\underline{\theta}&=(\underline{\theta}_1',\dots,\underline{\theta}_K')'=\left(\left(\underline{\varphi}_1',\underline{\alpha}_1'\right);\dots;\left(\underline{\varphi}_K',\underline{\alpha}_K'\right)\right)'\\&=((\varphi_{1,1},...,\varphi_{1,p},\underline{\alpha}_1');\dots;(\varphi_{K,1},...,\varphi_{K,p},\underline{\alpha}_K'))'\in(0,1)^{Kp}\times A^K\subset\mathbb{R}_+^{K(p+q)}\end{aligned}$$

The parametric distribution with parameters $\varphi_{j,i}\in(0,1)$,$i=1,\dots,p$, is denoted by $\mathbb{F}_{\varphi_{j,i}}$, and $f_{\varphi_{j,i}}$ denotes its mass function. In general, we note a probability measure on $\mathbb{Z}_+$ by a capital, and its probability mass function by the corresponding lower letter. This implies that $g_{\underline{\alpha}_j}$ is the point mass function of the discrete distribution $\mathbb{G}_{\underline{\alpha}_j}$ for j being fixed in $\{1,\dots,K\}$. $\mathcal{G}$ indicates the set of all probability measures on $\mathbb{Z}_+=\mathbb{N}\cup\{0\}$. For $\mathbb{G}_j\in\mathcal{G}$, $\mu_{\mathbb{G}_j}$ and $\sigma^2_{\mathbb{G}_j}\in[0,+\infty]^2$ represent the mean and the variance of $\mathbb{G}_j$ respectively. For probability measures $\mathbb{F}_j$ and $\mathbb{G}_j$, $\mathbb{F}_j\circledast\mathbb{G}_j$ indicates the convolution of $\mathbb{F}_j$ and $\mathbb{G}_j$, with fixed $j\in\{1,\dots,K\}$. $\{\mathcal{F}_t\}_{t\geq1-p}$ is the natural filtration generated by $y_{1-p},\dots,y_t$, i.,e., $\mathcal{F}_t=\sigma\left(y_{1-p},\dots,y_t\right)$.

2 Preliminary probabilistic results

In this section, we discuss the basic probabilistic and statistical properties of the new self-exciting threshold $GINAR\,(p)$ model with multiple K regimes, $K\geq2$, $(SET-GINAR\,(K,p))$. First, we annunciate the following proposition which proves that there exists a strictly stationary $SET-GINAR\,(K;p)$-model satisfying (1.1).

Proposition 2.1. *There exists a unique strictly stationary solution satisfying* (1.1) with $Cov\,(y_s,\varepsilon_{j,t})=0$ *for* $j=1,\dots,K$ *and all* $s<t$, *if the spectral radius of the following matrix*

$$\Phi_{\max}=\begin{pmatrix}\varphi_{\max,1} & \varphi_{\max,2} & \cdots & \varphi_{\max,p-1} & \varphi_{\max,p}\\ 1 & 0 & \cdots & 0 & 0\\ 0 & 1 & \ddots & & \vdots\\ \vdots & & \ddots & 0 & \\ 0 & & & 1 & 0\end{pmatrix}$$

is lesser than 1. *In addition,* $\{y_t, t \in Z\}$ *is an ergodic process.*
Second, we compute the first two unconditional moments of the $SET-GINAR(K;p)$ model, to get an idea on the moment structure of the model (then we can use these results to obtain moments-based estimators). Proposition (2.2) below gives the mean and the variance for the process given by (1.1).
Proposition 2.2. *Putting the following notations:* $p_j^* = P(y_{t-d} \in R_j)$, $u_{j,i} = \mathbb{E}(y_{t-i}|y_{t-d} \in R_j)$, $\sigma_{j,i}^2 = \mathbb{V}ar(y_{t-i}|y_{t-d} \in R_j)$, $u_j^{(n,m)} = \mathbb{E}(y_{t-n}y_{t-m}|y_{t-d} \in R_j)$, *and* $u_{k,l}^* = \mathbb{E}\left(\mathbf{1}_{(S_t=k)}\mathbf{1}_{(S_t=l)}\right)$.

1) *The mean of* y_t *is given by:*
$$\mathbb{E}_{\underline{\theta}}[y_t] = \sum_{j=1}^K p_j^* \left[\sum_{i=1}^p \varphi_{j,i}u_{j,i} + \mu_{\mathbb{G}_{\underline{\alpha}_j}}\right]. \tag{2.3}$$
2) *The variance of* y_t *is given by:*
$$\mathbb{V}ar_{\underline{\theta}}[y_t] = \sum_{j=1}^K I_j + 2\underset{k<l,k,l=1,\dots,K}{\sum_k \sum_l} II_{k,l} \tag{2.4}$$
where
$$\begin{cases} I_j = p_j^*\left(\sigma_{\mathbb{G}_{\underline{\alpha}_j}}^2 + \sum_{i=1}^p \varphi_{j,i}\beta_{j,i}u_{j,i}\right) + \sum_{i=1}^p \varphi_{j,i}\left(\sigma_{j,i}^2 + u_{j,i}^2\right) - p_j^{*2}\varphi_{j,i}^2 u_{j,i}^2 \\ \qquad + 2\underset{n<m,n,m=1,\dots,p}{\sum_n \sum_m} \varphi_{j,n}\varphi_{j,m}p_j^*\left(u_j^{(n,m)} - p_j^* u_{j,n}u_{j,m}\right) \\ II_{k,l} = 1_{k,l} + 2_{k,l} + 3_{k,l} + 4_{k,l} \end{cases}$$
with
$$\begin{cases} 1_{k,l} = -p_k^* p_l^* \sum_{i=1}^p \varphi_{k,i}u_{k,i}\sum_{i=1}^p \varphi_{l,i}u_{l,i} & 3_{k,l} = \mu_{\mathbb{G}_{\underline{\alpha}_k}}\sum_{i=1}^p \varphi_{l,i}u_{l,i}\left(u_{k,l}^* - p_k^* p_l^*\right) \\ 2_{k,l} = \mu_{\mathbb{G}_{\underline{\alpha}_l}}\sum_{i=1}^p \varphi_{k,i}u_{k,i}\left(u_{k,l}^* - p_k^* p_l^*\right) & 4_{k,l} = \mu_{\mathbb{G}_{\underline{\alpha}_k}}\mu_{\mathbb{G}_{\underline{\alpha}_l}}\left(u_{k,l}^* - p_k^* p_l^*\right) \end{cases}$$
Remark 2.1. The condition $\sum_{i=1}^p \varphi_{j,i} < 1$ for a fixed $j = 1, \dots, K$, is a sufficient condition for both the second-order and strict stationarity, and the ergodicity of our model. It is easy to see that the above assumptions satisfy for example the often used distributions in the literature with $\mathbb{S} = (0, +\infty)$ and $\mathbb{G}_{\underline{\alpha}} = Poisson(\alpha)$ or $\mathbb{G}_{\underline{\alpha}} = Geometric(\alpha)$.

3 Parameters estimation issue

We consider the parameters estimation problem of the $SETGINAR(K;p)$ model, using two estimation methods, namely the conditional least squares method (CLS), and conditional maximum likelihood method (CML) methods. For the rest, we put $\mathbf{1}(S_t = j) = \mathbb{I}_{t-d,j}$.

3.1 Conditional least squares method

Let $\underline{\theta} = \left(\underline{\theta}_1', \underline{\theta}_2', \dots, \underline{\theta}_K'\right)'$ the $(p+q)K-$vector of model (2.1) parameters, and let $\underline{\theta}^* = \left(\underline{\theta}_1^{*\prime}, \underline{\theta}_2^{*\prime}, \dots, \underline{\theta}_K^{*\prime}\right)'$ the $(p+1)K-$vector of quasi parameters of the model (1.1). So, we have
$$\underline{\theta}^* = \left(\left(\varphi_{1,1}, \dots, \varphi_{1,p}, \mu_{\mathbb{G}_{\underline{\alpha}_1}}\right), \dots, \left(\varphi_{K,1}, \dots, \varphi_{K,p}, \mu_{\mathbb{G}_{\underline{\alpha}_K}}\right)\right)',$$
and by putting the quantity:
$$q\left(\underline{\theta}^*, \mathcal{F}_{t-1}\right) = \mathbb{E}_{\underline{\theta}^*}[y_t|y_{t-1}, \dots, y_{t-p}] = \sum_{j=1}^K \left[\left(\sum_{i=1}^p \varphi_{j,i}y_{t-i}\right) + \mu_{\mathbb{G}_{\underline{\alpha}_j}}\right]\mathbb{I}_{t-d,j}.$$
Then the $(p+1)K-$vector of CLS-estimators $\widehat{\underline{\theta}^*} = \left(\left(\widehat{\underline{\varphi}}_{1,n}', \widehat{\mu}_{\mathbb{G}_{\underline{\alpha}_{1,n}}}\right), \dots, \left(\widehat{\underline{\varphi}}_{K,n}', \widehat{\mu}_{\mathbb{G}_{\underline{\alpha}_{K,n}}}\right)\right)'$ are derived by minimizing the quantity:
$$Q_n\left(\underline{\theta}^*\right) = \sum_{t=1}^n \left(y_t - q\left(\underline{\theta}^*, \mathcal{F}_{t-1}\right)\right)^2 = \sum_{t=1}^n U_t^2\left(\underline{\theta}^*\right).$$
By solving the system of score equations arising from the partial derivatives of first order, we obtain the CLS-estimators $\widehat{\underline{\theta}^*} = \left\{\left(\widehat{\underline{\varphi}}_{j,n}', \widehat{\mu}_{\mathbb{G}_{\underline{\alpha}_{j,n}}}\right)'\right\}_{j=1,\dots,K}$.
To study the strong consistency and asymptotic normality of the CLS estimators, we make the following assumptions about the existence of the moments of the stationary distribution of the underlying $SET-GINAR(K,p)$ process.

Assumption $(A.1)$. The observed sequence $\{y_t, t \in \mathbb{Z}_+\}$ is generated from $SET-GINAR\,(K,p)$ process with the true parameters

$$\underline{\theta}_0 = \left(\underline{\theta}'_{1,0}, \dots, \underline{\theta}'_{K,0}\right)' = \left(\left(\underline{\varphi}'_{1,0}, \underline{\alpha}'_{1,0}\right); \dots; \left(\underline{\varphi}'_{K,0}, \underline{\alpha}'_{K,0}\right)\right)' \in (0,1)^{Kp} \times A^K \subset \mathbb{R}_+^{K(p+q)}.$$

Assumption $(A.2)$. Let $\mathbb{G}_{\underline{\alpha}_j} \in \mathcal{G}$ with $g_{\underline{\alpha}_j}(0) \in [0,1)$, $\mu_{\mathbb{G}_{\underline{\alpha}_j}} < \infty$, and $\underline{\varphi} \in (0,1)^{Kp}$ with $\sum_{i=1}^{p} \varphi_{j,i} < 1$ for $j \in \{1, \dots, K\}$ being fixed. Then, for $r \in \mathbb{N}^*$, $\mathbb{E}_{\underline{\theta}^*}(y_t^r) < \infty$, for $i, j, k = 1, \dots, p$, $i \neq j \neq k$, $\mathbb{E}_{\underline{\theta}^*}\left(y_{t-i}^2 y_{t-j}\right) < \infty$, $\mathbb{E}_{\underline{\theta}^*}\left(y_{t-i} y_{t-j} y_{t-k}\right) < \infty$.

Proposition 3.1. *Assuming that* $(A.1)-(A.2)$ *hold. Let for* $j \in \{1, \dots, K\}$, $\underline{\varphi}_{j,0} \in (0,1]^p$, *and* $\mu_{\mathbb{G}_{\underline{\alpha}_j,0}} \in \mathbb{R}_+$ *a"truth" autoregression parameter and a "truth" mean parameter,* υ *probability measure on* $\mathbb{Z}_+$ *with finite support, and* $\mathbb{G}_{\underline{\alpha}_j}$ *such that* $\mathbb{E}_{\mathbb{G}_{\underline{\alpha}_j}}[\varepsilon_1]^3 < \infty$ *and* $g_{\underline{\alpha}_j}(0) \in (0,1)$. *Then we have:*

$i)$ *Strong consistency of* CLS*-estimators:*

Any CLS*-estimators* $\widehat{\underline{\theta}}^*{}_n = \left(\widehat{\underline{\varphi}}'_{1,n}, \widehat{\mu}_{\mathbb{G}_{\underline{\alpha}_{1,n}}}, \dots, \widehat{\underline{\varphi}}'_{K,n}, \widehat{\mu}_{\mathbb{G}_{\underline{\alpha}_{K,n}}}\right)'$ *is consistent in the following sense:*

$$\widehat{\underline{\theta}}^*{}_n \xrightarrow{a.s} \underline{\theta}_0^* \Rightarrow \widehat{\underline{\varphi}}_{j,n} \xrightarrow{a.s} \underline{\varphi}_{j,0}, \text{ and } \widehat{\mu}_{\mathbb{G}_{\underline{\alpha}_{j,n}}} \xrightarrow{a.s} \mu_{\mathbb{G}_{\underline{\alpha}_{j,0}}}$$

$ii)$ *Asymptotic normality of* CLS*-estimators:*

$$\left(\sqrt{n}\left(\widehat{\underline{\theta}}^*{}_n - \underline{\theta}_0^*\right)'\right)' \xrightarrow{d} N_{(p+1)K}\left(0, V^{-1}WV^{-1}\right)$$

Where V *and* W *are square matrices of order* $(p+1)\,K$ *with following elements:*

$$V_{k,l} = \mathbb{E}_{\underline{\theta}^*}\left(\frac{\partial}{\partial \underline{\theta}_k^*} q\left(\underline{\theta}^*, \mathcal{F}_{t-1}\right) \frac{\partial}{\partial \underline{\theta}_l^*} q\left(\underline{\theta}^*, \mathcal{F}_{t-1}\right)\right)$$

$$W_{k,l} = \mathbb{E}_{\underline{\theta}^*}\left(U_t^2\left(\underline{\theta}^*\right) \frac{\partial}{\partial \underline{\theta}_k^*} q\left(\underline{\theta}^*, \mathcal{F}_{t-1}\right) \frac{\partial}{\partial \underline{\theta}_l^*} q\left(\underline{\theta}^*, \mathcal{F}_{t-1}\right)\right)$$

Such as $\underline{\theta}_i^*$ *denotes the* i*th component of* $\underline{\theta}^*$.

3.2 Conditional maximum likelihood method

In the most case, the maximum likelihood estimation method is not directly applicable in integer-valued time series models. Here, the conditional maximum likelihood is used, since closed expressions for $\upsilon_0\left(\underline{\theta}\right)$ are only known for a few specific immigration distribution. We call an estimator $\left(\widehat{\underline{\theta}}_n\right)_{n \in \mathbb{Z}_+}$ of $\underline{\theta}$ a conditional maximum likelihood CML estimator of $\underline{\theta}$ if $\widehat{\underline{\theta}}_n$ maximizes the conditional likelihood function associated to the model, i.e.,

$$\forall n \in \mathbb{Z}_+ : \left(\widehat{\underline{\theta}}_n\right) \in \underset{\left(\underline{\theta}\right) \in (0,1]^{Kp} \times A^K}{\arg} \max\left(\prod_{t=1}^{n} P^{\underline{\theta}}_{(y_{t-1}, \dots, y_{t-p}), y_t}\right),$$

More precisely, we have the following transition probability in terms of probability masses:

$$P^{\underline{\theta}}_{(y^*_{t-1}, \dots, y^*_{t-p}), y^*_t} = \sum_{j=1}^{K}\left[\sum_{k_1 + \dots k_p} \sum_{k_1} \dots \sum_{k_p}\left(\left[f_{\varphi_{j,1}}(k_1) \dots f_{\varphi_{j,p}}(k_p)\right] \times g_{\underline{\alpha}_j}\left(y_t^* - (k_1 + \dots k_p)\right)\right)\right] \mathbb{I}_{t-d,j}$$

We emphasize that we, nowhere, impose that such a maximum location is unique. The following Proposition states the consistency and asymptotic normality of the CML-estimators.

Proposition 3.2. *Let* $\underline{\varphi}_{j,0} \in (0,1]^p$, *and* $\underline{\alpha}_{j,0} \in A$ *a"truth" autoregression parameter and a "truth" innovation parameter,* υ *probability measure on* $\mathbb{Z}_+$ *with finite support, and* $\mathbb{G}_{\underline{\alpha}_j}$ *such that* $\mathbb{E}_{\mathbb{G}_{\underline{\alpha}_j}}[\varepsilon_1]^3 < \infty$ *and* $g_{\underline{\alpha}_j}(0) \in (0,1)$ *for all* $j \in \{1, \dots, K\}$ *being fixed. Then, under the Franke and Seligmann* (1993) *conditions* $(C.1)-(C.6)$, *we have:*

$i)$ *Consistency of* CML*-estimators:*

Any CML*-estimators* $\widehat{\underline{\theta}}_n = \left(\widehat{\underline{\varphi}}'_{1,n}, \widehat{\underline{\alpha}}'_{1,n}, \dots, \widehat{\underline{\varphi}}'_{K,n}, \widehat{\underline{\alpha}}'_{K,n}\right)'$ *is consistent in the following sense:*

$$\widehat{\underline{\theta}}_n \xrightarrow{p} \underline{\theta}_0 \Rightarrow \widehat{\underline{\varphi}}_{j,n} \xrightarrow{p} \underline{\varphi}_{j,0}, \text{ and } \widehat{\underline{\alpha}}_{j,n} \xrightarrow{p} \underline{\alpha}_{j,0}$$

ii) Asymptotic normality of CML-estimators:

$$\left(\sqrt{n}\left(\widehat{\underline{\theta}}_n-\underline{\theta}_0\right)'\right)' \xrightarrow{d} N_{K(p+q)}\left(0,\Gamma\left(\underline{\theta}\right)^{-1}\right)$$

Where $\Gamma\left(\underline{\theta}\right)$ is Fisher information matrix of order $K\left(p+q\right)$.

4 Nonlinearity testing problem

Assume that $K=2$, the present paragraph contributes to constructing while following the Le Cam (1960) methodology, an efficient parametric test of the existence of a threshold effect in the $GINAR\left(p\right)$ process. We generalize the recent test of nonlinearity for a threshold in $INAR\left(p\right)$ model which is proposed by Yang *et al* (2022), while adopting a perturbation of the parameters introducing a perturbation due to the regime change.
The following proposition establishes a locally asymptotically optimal test (so-called most stringent test) to test a time-invariant $GINAR\left(p\right)$ model, against a $SET-GINAR(2,p)$ model given by (1.1).

Proposition 4.1. (*The most stringent test*). *Under some regularity conditions, the test rejecting the null hypothesis $H_g^{(n)}\left(\underline{\theta}\right)$ whenever:*

$$\widehat{Q}_g^{(n)}\left(\widehat{\underline{\theta}}^{(n)}\right)=\widehat{\underline{\Delta}}^{(n)\prime}\left(\widehat{\underline{\theta}}^{(n)}\right)\left(\Gamma^{\underline{\Delta}^{(n)}}\left(\widehat{\underline{\theta}}^{(n)}\right)\right)^{-1}\widehat{\underline{\Delta}}^{(n)}\left(\widehat{\underline{\theta}}^{(n)}\right)>\mathcal{X}^2_{2(p+q),1-\alpha},$$

is such that :

(i) has asymptotic level α under $H_g^{(n)}\left(\underline{\theta}\right)$,

(ii) has asymptotic power:

$$1-\left(\mathcal{X}^2_{1-\alpha};\left(p+q\right);\underline{\tau}_2^{*\prime}\Gamma\left(\widehat{\underline{\theta}}^{(n)}\right)\underline{\tau}_2^{*}\right),\ \text{under } H_g^{(n)}\left(\underline{\theta}+\underline{\nu}^{(n)}\underline{\tau}^{*(n)}\right),$$

where $\left(\mathcal{X}^2_{1-\alpha};r;v\right)$ denotes the non central chi-square distribution function with r degrees of freedom and non-centrality parameter v;

(iii) is locally asymptotically most stringent test against $H_g^{(n)}\left(\underline{\theta}+\underline{\nu}^{(n)}\underline{\tau}^{*(n)}\right)$.

5 Numerical illustrations

Our illustrations have been structured into two main parts: an intensive simulation study and a real data example. We numerically assessed our statistical procedures first on artificial data generated by our own algorithms, then applied our theoretical results to a real dataset. We note that in this document we didn't report the real data application.

5.1 Simulation results

We have evaluated the (CLS), and the (CML) estimators, on several time series, of small, moderate, and relatively large sizes n. In this paragraph, we consider series were generated from three different $SET-GINAR\left(3,2\right)$ models, based on three different thinning operators (Binomial: $\varphi*y|y\mapsto\mathcal{B}in\left(y,\varphi\right)$, Poisson: $\varphi*y|y\mapsto\mathcal{P}\left(\varphi y\right)$, and Negative-Binomial: $\varphi*y|y\mapsto\mathcal{NB}\left(y,1/1+\varphi\right)$). Data-generating models are driven by Poisson $\mathcal{P}\left(\alpha_j\right)$, and Geometric $\mathcal{G}\left(e^{-x}\equiv\alpha_j\right)$ innovation distributions, which are considered in Model $M1$ and Model $M2$, respectively. The choice $\alpha=e^{-x}\in(0,1]$ with $x\geq 0$ for the $\mathcal{G}$ distribution is mentioned in Drost *et al* (2009) due to the same masses assignment to 0. The models are given as follows:

$$\begin{aligned}\text{Model } M1:\underline{\theta}&=\left[(\varphi_{1,1},\varphi_{1,2},\alpha_1;\varphi_{2,1},\varphi_{2,2},\alpha_2;\varphi_{3,1},\varphi_{3,2},\alpha_3;[c_1,c_2])\right]'\\&=\left[(0.3,0.6,2;0.15,0.2,1;0.5,0.1,3;[6,11])\right]'.\ \mathcal{P}\left(\alpha_j\right),j=1,2,3\\\text{Model } M2:\underline{\theta}&=\left[(\varphi_{1,1},\varphi_{1,2},\alpha_1;\varphi_{2,1},\varphi_{2,2},\alpha_2;\varphi_{3,1},\varphi_{3,2},\alpha_3;[c_1,c_2])\right]'\\\underline{\theta}&=\left[(0.3,0.6,2;0.15,0.2,1;0.9,0.1,3;[6,11])\right]'.\ \mathcal{G}\left(e^{\alpha_j}\right),j=1,2,3\end{aligned}$$

We stress that we report in this document only the results covering the Poisson $\mathcal{P}\left(\alpha_j\right)$ innovation distributions. Our numerical results concerning the estimation problem are summarized in Table 1 given below including only the CLS estimators.

Table 1. Simulation results of the CLS estimates for Model $M1$ with delay $d = 1$

Method	n	Result	$\varphi_{1,1}$	$\varphi_{1,2}$	α_1	$\varphi_{2,1}$	$\varphi_{2,2}$	α_2	$\varphi_{3,1}$	$\varphi_{3,2}$	α_3
$* \mapsto \mathcal{B}in$	100	$Mean$	.2806	.5723	4.3016	.1618	.1943	.9364	.4212	.0464	4.2028
		$RMSE$	(.3161)	(.1794)	(1.888)	(.1602)	(.0666)	(1.477)	(.5987)	(.2164)	(7.761)
	300	$Mean$	.2835	.5855	4.1976	.1473	.1991	1.0186	.4793	.0864	3.3512
		$RMSE$	(.1661)	(.1099)	(1.116)	(.0840)	(.0324)	(.7463)	(.2753)	(.0772)	(3.662)
	800	$Mean$	.3065	.5921	4.0507	.1490	.1993	1.0169	.4975	.0956	3.1744
		$RMSE$	(.1051)	(.0635)	(.6628)	(.0554)	(.0212)	(.5025)	(.1428)	(.0448)	(1.962)
$* \mapsto \mathcal{P}$	100	$Mean$	.2756	.5707	4.3477	.1514	.1883	1.0521	.4584	.0433	3.7794
		$RMSE$	(.3797)	(.2253)	(2.266)	(.2020)	(.0745)	(1.714)	(.5203)	(.1747)	(6.860)
	300	$Mean$	.2832	.5860	4.1985	.1457	.2014	1.0434	.4558	.0868	3.6597
		$RMSE$	(.2035)	(.1131)	(1.1687)	(.1093)	(.0391)	(.9531)	(.2310)	(.0818)	(3.189)
	800	$Mean$	.2973	.5980	4.0373	.1489	.1998	.9557	.4936	.0964	3.0934
		$RMSE$	(.1284)	(.0743)	(.7744)	(.0640)	(.0220)	(.5537)	(.1284)	(.0512)	(1.784)
$* \mapsto \mathcal{NB}$	100	$Mean$	.2475	.5591	4.5384	.1418	.1934	1.1141	.4404	.0529	4.1234
		$RMSE$	(.3970)	(.2247)	(2.328)	(.2091)	(.0712)	(1.770)	(.4029)	(.1834)	(5.803)
	300	$Mean$	.3178	.5946	4.0323	.1541	.1972	.9648	.4795	.0804	3.9533
		$RMSE$	(.2353)	(.1200)	(1.305)	(.1098)	(.0414)	(.9269)	(.2268)	(.0839)	(3.244)
	800	$Mean$	.2938	.6009	4.0438	.1500	.1989	1.0108	.4936	.0932	3.4441
		$RMSE$	(.1412)	(.0730)	(.7711)	(.0712)	(.0235)	(.6035)	(.1265)	(.0470)	(1.864)

For the testing problem, we consider two $SET - GINAR(2,2)$ data-generator processes, $M1$, $M2$, and $GINAR(2)$ process $M3$ are used to simulate time series of small, moderate and relatively large sizes ($n = 100 - 1000$). $[M1, M2,\ M3]$ are based on three different thinning operators. Data-generating models are driven only by Poisson $\mathcal{P}(\alpha_j)$ innovation distributions, and $d = 1$.

Model $M3 : \underline{\theta} = [(\varphi_{1,1}, \varphi_{1,2}, \alpha_1; \varphi_{2,1}, \varphi_{2,2}, \alpha_2; c)]' = [(0.1, 0.5, 6; 0.6, 0.2, 3; 13)]'$.

Model $M4 : \underline{\theta} = [(\varphi_{1,1}, \varphi_{1,2}, \alpha_1; \varphi_{2,1}, \varphi_{2,2}, \alpha_2; c)]' = [0.1, 0.3, 4; 0.15, 0.2, 3; 6]$.

Model $M5 : \underline{\theta} = [(\varphi_1, \varphi_2, \alpha_1)]' = [0.1, 0.5, 3]$

The empirical powers of models $M1$ and $M2$, and also empirical type I errors for model $M3$ under confidence level of 95% when H_0 is true, are given in Table 2.

Table 2. Empirical powers and levels of the efficient tests ϕ for the level 5%

	n	100	200	300	400	600	800	1000
		ϕ	ϕ	ϕ	ϕ	ϕ	ϕ	ϕ
$M3$	$* \mapsto \mathcal{B}in$	.9600	.9967	1	1	1	1	1
	$* \mapsto \mathcal{P}$	.9533	.9960	1	1	1	1	1
	$* \mapsto \mathcal{NB}$	.9067	.9933	1	1	1	1	1
$M4$	$* \mapsto \mathcal{B}in$	.7000	.9433	.9760	.9911	1	1	1
	$* \mapsto \mathcal{P}$	.6300	.9400	.9733	1	1	1	1
	$* \mapsto \mathcal{NB}$	.6667	.9533	.9873	.9967	1	1	1
$M5$	$* \mapsto \mathcal{B}in$	.0300	.0590	.0431	.0600	.0500	.0533	.0487
	$* \mapsto \mathcal{P}$	.0630	.0350	.0567	.0400	.0477	.0461	.0493
	$* \mapsto \mathcal{NB}$	.0433	.0333	.0467	.0603	.0533	.0500	.0500

References

[1] Al-Osh, M. and Alzaid, A.A. (1987). First order integer-valued autoregressive ($INAR(1)$) processes. *Journal of Time Series Analysis* $8 : 261 - 275$.

[2] Bentarzi, M. and Sadoun, M. (2021). Efficient estimation in semiparametric self-exciting threshold $INAR$ processes. *Communications in Statistics-Simulation and Computation.* DOI: 10.1080/03610918.2021.1910300.

[3] Doukhan, P. Latour, A. and Oraichi, D. (2006). A Simple Integer-Valued Bilinear Time Series Model. *Adv. Appl. Prob.* $38(2) : 559 - 578$.

[4] Du, J-G and Li, Y. (1991). The integer-valued autoregressive ($INAR(p)$) model. *Journal of Time Series Analysis.* $12(2) : 129 - 142$.

[5] Drost, C. van den Akker, R. and Werker, B. (2009). Efficient estimation of auto-regression parameters and innovation distributions for semiparametric integer-valued $AR(p)$ models. *J. R. Statist. Soc. B* 71, *Part* 2, *pp.* $467-485$. Latour, A. (1997). The Multivariate $GINAR(p)$ Process. *Adv. Appl. Prob.* $29.228-248$.
[6] Le Cam, L. (1960). Locally asymptotically normal families of distributions. *University of California Publications in Statistics.* $3:37-98$.
[7] Monteiro, M. Scotto, M.G. and Pereira, I. (2012). Integer-Valued Self-Exciting Threshold Autoregressive process. *Communication in statistics-Theory and Methods* $41:2717-2737$.
[8] Thyregod, P., Carstensen, J., Madsen, H., and Arnbjerg-Nielsen, K. (1999). Integer valued autoregressive models for tipping bucket rainfall measurements. *Environmetrics*, $10,395-411$.
[9] Yang, K., Wang, D., Jia, B., & Li, H. (2018). An integer-valued threshold autoregressive process based on negative binomial thinning. *Statistical Papers*, 59 (3), $1131-1160$.
[10] Yang, k., Li, A., Li, H., and Dong, X. (2022). High-Order Self-excited Threshold Integer-Valued Autoregressive Model: Estimation and Testing. *Communications in Mathematics and Statistics*. $Doi.org/10.1007/s40304-022-00325-3$.