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Consistency of a geometric-type estimator for tail index under weak dependence

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Abstract

The aim of this paper is the estimation of the tail index parameter β under weak dependence in the sense of Doukhan [5]. This notion of weak dependence is more general than the classical frameworks of mixing. In the situation of mixing, the consistency of the geometric-type estimator was investigated by Brito and Freitas [3]. Here relaxing this requirement by a weak dependence assumption and we provide an application of this result for infinite moving average with heavy tail innovations.

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1 Introduction

Let $X_1, X_2, \ldots, X_n$ be a sequence of random variables with common distribution function F and denote $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$ the corresponding order statistics. We consider the problem of estimating the heavy tail index of a distribution F, with a tail function $\overline{F} = 1 - F$ verifying the following condition:

$$\overline{F}(x) = x^{-\beta} L(x), \; x > 0, \; \beta > 0 \tag{1}$$

L is a slowly varying function.

Tail index estimation is a central topic in extreme value analysis; it has received considerable attention. Several estimators for tail index have been proposed in the literature (see e.g. Dekkers et al. [4], Bacro and Brito [1] and references therein).One of the most commonly used estimators is that proposed by Hill [6], the Hill estimator of the shape parameter $\dfrac{1}{\beta}$ based on the k-largest observations is given by :

$$H_{(k_n)} = \frac{1}{k} \sum_{i=1}^{k} \log \frac{X_{(i)}}{X_{(k+1)}}$$

where $k = k_n$ and $(k_n)_n$ is an intermediate sequence that is,

$$k_n \to \infty, \; \frac{k_n}{n} \to 0 \; \text{ as } n \to \infty \tag{2}$$

Based on least squares considerations, Schultze and Steinebach [8] proposed three estimators, denoted by $\widehat{\beta}_1(k_n)$,$\widehat{\beta}_2(k_n)$ and $\widehat{\beta}_3(k_n)$, for the tail coefficient β, in the i.i.d. case. Brito and Freitas [2] have introduced a geometric-type estimator of β,$\widehat{\beta}(k_n)$, which is related to the least square estimators $\widehat{\beta}_1(k_n)$ and $\widehat{\beta}_3(k_n)$ and defined by

$$\hat{\beta} = \sqrt{\frac{\sum_{i=1}^{k} \log^2(\frac{n}{i}) - \frac{1}{k}\left(\sum_{i=1}^{k} \log(\frac{n}{i})\right)^2}{\sum_{i=1}^{k} X_{(n-i+1)}^2 - \frac{1}{k}\left(\sum_{i=1}^{k} X_{(n-i+1)}\right)^2}}$$

2 Preliminaries

2.1 Weak dependence

Define by $\mathbb{L}^\infty$ the union of the sets $\mathbb{L}^u, u \in \mathbb{N}^*$ of numerical bounded measurable functions on some Euclidean space $\mathbb{R}^u$ and $\|.\|_\infty$ the corresponding uniform norm. We define the Lipschitz modulus of a function $h : \mathbb{R}^u \to \mathbb{R}$ by

$$Lip(h) = \sup_{x \neq y} \frac{|h(x) - h(y)|}{\|x - y\|_1}$$

where $\|x - y\|_1 = \sum_{i=1}^n |x_i - y_i|$.
Consider the class $\mathcal{L} = \{h \in \mathbb{L}^\infty; \|h\|_\infty \le 1, Lip(h) < \infty\}$.

▶ **Definition 1.** *A process $X = (X_n)_n$ is called (ε, ψ)-weakly dependent process if there exist a function $\psi : (\mathbb{N}Í*)^2 \times (\mathbb{R}^+)^2 \to \mathbb{R}^+$ and a sequence $(\varepsilon_r)_r \in \mathbb{N}$ such that $\varepsilon_r \to 0$ as $r \to \infty$ satisfying,*
$\forall u, v \in \mathbb{N}^*$, $\forall i_1 < .... < i_u \le i_u + r \le j_1 < .... < j_v$,
$\forall h, g \in \mathcal{L}, h : \mathbb{R}^u \to \mathbb{R}; g : \mathbb{R}^v \to \mathbb{R}$,

$$| \; Cov(h(X_{i_1}, ..., X_{i_u}), g(X_{j_1}, ..., X_{j_v})) \; | \le \psi(u, v, Liph, Lipg)\varepsilon_r$$

2.2 Consistency of geometric-type estimator in the strongly mixing case

In this part, we recall Brito and Frietas result concerning the consistency of the geometric-type estimator, under the condition of strong mixing.
The strong mixing coefficient for a stationary sequence $\{\xi_i\}$ is defined by:
$\alpha_l(\xi_i) = \sup\{| \; P(A \cap B) - P(A)P(B) \; |: A \in \mathcal{F}_1^j\{\xi_i\}, B \in \mathcal{F}_{j+l+1}^\infty\{\xi_i\}, j > 1\} \to 0$ as $l \to \infty$, where $\mathcal{F}_r^s$ denotes the σ-field $\sigma\{\xi_i : r \le i \le s\}$.
We begin by considering the following functionals of X_i;

$$Y_{ni} = \Big(\log X_i - \log b(\frac{n}{k})\Big)_+ \quad \text{and } I_{ni} = 1_{\{\log X_i - \log b(\frac{n}{\rho k}) > \epsilon\}} \; , \epsilon \in \mathbb{R}$$

where ρ is in some interval containing 1, for $x \in \mathbb{R}$, x_+ indicates $\max(x, 0)$.
Denote by $b(t) = F^{\leftarrow}(1 - \frac{1}{t})$, $t > 1$, the quantile function where, $F^{\leftarrow}(t) = inf(x : F(x) \ge t, \; 0 < t < 1)$ the left-continuous inverse function of F.

▶ **Theorem 2.** *[Brito and Frietas [3]] Suppose F satisfies (1) and k_n be a sequence of positive integers such that (2) holds.*
If,

$$\frac{1}{k_n} \sum_{i=1}^{n} (T_{ni} - E(T_{ni})) \xrightarrow{P} 0 \tag{3}$$

for $T_{ni} = Y_{ni}^j, j = 1, 2$ and $T_{ni} = I_{ni}$.
Then $\widehat{\beta}$ converge in probability to β.

Brito and Frietas proves that convergence in probability given in (3), can be deduced from the convergence of sequence $(S_{nj})_j$ defined by:

$$S_{nj}(T_{ni}) = \sum_{i=(j-1)l_n+1}^{jl_n} T_{ni}$$

consider the following conditions

$$\begin{cases} \frac{1}{k} \sum_{i=1}^{r_n} E(|S_{nj}| 1_{|S_{nj}| > k}) \underset{n \to +\infty}{\to 0} \\ \frac{1}{k^2} \sum_{i=1}^{r_n} E(S_{nj}^2 1_{|S_{nj}| \le k}) \underset{n \to +\infty}{\to 0} \end{cases} \tag{4}$$

▶ **Theorem 3.** *[Brito and Frietas [3]] Let $(X_i)_i$ be a strictly stationary sequence and k_n a sequence of positive integers satisfying (2). Assume that,*
(i) for $T_{ni} = Y_{ni}$ and $T_{ni} = I_{ni}$, there exists a sequence l_n of positives integers such that $\frac{l_n}{n} \to 0$, $r_n \alpha_{n,l_n}(T_{ni}) \underset{n\to+\infty}{\to 0}$ and the conditions (4) is satisfied for $S_{nj}(T_{ni})$.
(ii) The conditions (4) is satisfied for $S_{nj}(Y_{ni}^2)$.
Then

$$\widehat{\beta} \xrightarrow{P} \beta \tag{5}$$

The consistency of the estimator $H_{(k_n)}$ of $\frac{1}{\beta}$ was established in Theorem 3.2 of Hsing [7] under the condition (i) of the previous theorem.

3 Main results

3.1 Consistency of a geometric-type estimator under weak dependence

We adapt the same method as Brito and Frietas to show the consistency of $\widehat{\beta}$ under weak dependence. First, we will extend the result of Theorem 3 for weakly dependent functionals.

▶ **Theorem 4.** *Let $(X_t)_t$ be a strictly stationary sequence and k_n a sequence of positive integers satisfying (2).*
If for $r_n = \left[\frac{n}{l_n}\right]$, we have

$$\lim_{n\to\infty} \frac{r_n}{k_n^2}(var(S_{n1})) = 0, \tag{6}$$

and

$$(T_{ni})_i \text{ for } T_{ni} = Y_{ni} \text{ and } T_{ni} = I_{ni} \text{ is } (\varepsilon_{l_n}, \psi)\text{-weakly dependent sequence}, \tag{7}$$

Then

$$\widehat{\beta} \xrightarrow{P} \frac{1}{\beta}$$

The mixing conditions imposed in Theorem 3 are replaced by the weak dependence condition, which is more flexible.

▶ **Theorem 5.** *Assume that F, satisfies (1), let $(X_t)_t$ be a strictly stationary sequence and k_n is a positive integers satisfying (2). Suppose that:*

$$(T_{ni})_i \text{ for } T_{ni} = Y_{ni} \text{ and } T_{ni} = I_{ni} \text{ is } (\varepsilon_{l_n}, \psi)\text{-weakly dependent sequence}, \tag{8}$$

and $\dfrac{l_n}{k_n} \to 0$ as $n \to \infty$
Then, the estimators $\widehat{\beta}$ and $\dfrac{1}{H}$ converge to β in probability.

3.2 Tail index estimation for infinite order moving averages

Let $(X_t)_t$ be a strictly stationary linear process defined by

$$X_t = \sum_{i\geq 0} c_i \varepsilon_{t-i} \tag{9}$$

$(\varepsilon_t)_t$ is an $i.i.d$ sequence of random variables with marginal distribution satisfying

$$\overline{G}(x) = 1 - G(x) = x^{-\beta} L(x),\ x > 0,\ \beta > 0 \tag{10}$$

L is a slowly varying function at infinity.
Under certain mild summability conditions of the sequence $(c_j)_j$, the condition (10) on the distribution of the $i.i.d$ innovations guaranteeing that the distribution of X_i be regularly varying.

▶ **Corollary 6.** *Let $(X_t)_t$ be a linear process given by (9) with common distribution F, satisfying assumptions (6), (10). We assume that there exists some constant $C > 0$, such that*

$$\|T_{ni}\|_p \leq C, \quad with \ \ p > 2 \tag{11}$$

Then,

$$\widehat{\beta} \xrightarrow{P} \frac{1}{\beta}$$

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